Tutorial 4 2022.10.19

Let $a_1, a_2, \cdots, a_n$ be n positive numbers.

We define

$$D(a_1, a_2, \cdots, a_n) := \{(x_1, \cdots, x_n) \in \mathbb{R}^n | x_1^{a_1} + \cdots + x_n^{a_n} \leq 1, \text{ and } x_i \geq 0, \forall i = 1, 2, \cdots, n\}.$$

Let's call $D(a_1, a_2, \cdots, a_n)$ a **spherical type domain** (named by me temporarily, since I don't know whether it has an official name). We are going to calculate the volume of $D(a_1, a_2, \cdots, a_n)$, denoted by $V(a_1, a_2, \cdots, a_n)$.

For example, $V(2, 2)$ is a quarter of the volume of the disk $\mathbb{B}^2 = \{(x, y) | x^2 + y^2 \leq 1\}$ and $V(2, 2, 2)$ is one-eighth of the volume of the ball $\mathbb{B}^3 = \{(x, y, z) | x^2 + y^2 + z^2 \leq 1\}$. That's why we call it spherical type.

You can obtain the volume of $\mathbb{B}^2$ and $\mathbb{B}^3$ immediately, but we will present here a unified approach to calculate $V(a_1, a_2, \cdots, a_n)$. To begin with, we need some extra ingredients.

4.1 Gamma function

The **Gamma function** Γ is a function defined over positive real numbers $\alpha \in \mathbb{R}_{>0}$.

$$\Gamma(\alpha) := \int_0^{+\infty} x^{\alpha-1} \mathrm{e}^{-x} \, \mathrm{d}x$$

We have obviously

$$\Gamma(1) = \int_0^{\infty} e^{-t} dt = 1$$

and for $x > 0$, an integration by parts yields

$$\Gamma(x+1) = \int_0^{\infty} t^x e^{-t} dt = \left[-t^x e^{-t}\right]_0^{\infty} + x \int_0^{\infty} t^{x-1} e^{-t} dt = x\Gamma(x),$$

and the relation

$$\Gamma(x+1) = x\Gamma(x) \tag{4.1}$$

is the important functional equation. For integer values the functional equation becomes

$$\Gamma(n+1) = n!,$$

and it's why the gamma function can be seen as an extension of the factorial function to real non null positive numbers.

The values of Gamma function at half integers can also be calculated:

The change of variable $t = u^2$ gives

$$\Gamma(1/2) = \int_0^{\infty} \frac{e^{-t}}{\sqrt{t}} dt = 2 \int_0^{\infty} e^{-u^2} du = 2\frac{\sqrt{\pi}}{2} = \sqrt{\pi}.$$

This is the Gaussian integral that we have met in assignment 3. The functional equation 4.1 entails for positive integers n

$$\Gamma\left(n + \frac{1}{2}\right) = \frac{1.3.5 \ldots (2n-1)}{2^n} \sqrt{\pi}, \tag{4.2}$$

We also have

Proposition 4.1

$$\frac{\Gamma(\alpha) \cdot \Gamma(\beta)}{\Gamma(\alpha+\beta)} = \int_0^1 x^{\alpha-1}(1-x)^{\beta-1} dx$$

Proof Let $x = \frac{y}{1+y}$, we have $\int_0^1 x^{\alpha-1}(1-x)^{\beta-1}dx = \int_0^{+\infty} \frac{y^{\alpha-1}}{(1+y)^{\alpha+\beta}}dy$. So

$$\begin{aligned}\Gamma(\alpha+\beta)\cdot\int_0^1 x^{\alpha-1}(1-x)^{\beta-1}dx &= \int_0^{+\infty} \frac{\Gamma(\alpha+\beta)y^{\alpha-1}}{(1+y)^{\alpha+\beta}}\mathrm{d}y = \\ &= \int_0^{+\infty}\left(y^{\alpha-1}\int_0^{+\infty} x^{\alpha+\beta-1}\mathrm{e}^{-(1+y)x}\,\mathrm{d}x\right)\mathrm{d}y \overset{!}{=} \\ &\overset{!}{=} \int_0^{+\infty}\left(\int_0^{+\infty} y^{\alpha-1}x^{\alpha+\beta-1}\mathrm{e}^{-(1+y)x}\,\mathrm{d}y\right)\mathrm{d}x = \\ &= \int_0^{+\infty}\left(x^{\beta-1}\mathrm{e}^{-x}\int_0^{+\infty}(xy)^{\alpha-1}\mathrm{e}^{-xy}x\,\mathrm{d}y\right)\mathrm{d}x = \\ &= \int_0^{+\infty}\left(x^{\beta-1}\mathrm{e}^{-x}\int_0^{+\infty}u^{\alpha-1}\mathrm{e}^{-u}\,\mathrm{d}u\right)\mathrm{d}x = \Gamma(\alpha)\cdot\Gamma(\beta).\end{aligned}$$

4.2 Dirichlet's integral[1]

We shall now show how the repeated integral

$$I = \iint\ldots\int f\left(t_1+t_2+\ldots+t_n\right)t_1^{\alpha_1-1}t_2^{\alpha_2-1}\ldots t_n^{\alpha_n-1}dt_1dt_2\ldots dt_n$$

may be reduced to a simple integral, where f is continuous, $\alpha_r > 0(r = 1,2,\ldots n)$ and the integration is extended over all positive values of the variables such that $t_1+t_2+\ldots+t_n \leqslant 1$. To simplify $\int_0^{1-\lambda}\int_0^{1-\lambda-T} f(t+T+\lambda)t^{\alpha-1}T^{\beta-1}dtdT$ (where we have written t, T, α, β for $t_1, t_2, \alpha_1, \alpha_2$ and λ for $t_3+t_4+\ldots+t_n$), put $t = T(1-v)/v$; the integral becomes (if $\lambda \neq 0$)

$$\int_0^{1-\lambda}\int_{T/(1-\lambda)}^1 f(\lambda+T/v)(1-v)^{\alpha-1}v^{-\alpha-1}T^{\alpha+\beta-1}dvdT.$$

Changing the order of integration, the integral becomes

$$\int_0^1\int_0^{(1-\lambda)v} f(\lambda+T/v)(1-v)^{a-1}v^{-a-1}T^{a+\beta-1}dTdv.$$

Putting $T = v\tau_2$ and by proposition 4.1, the integral becomes

$$\int_0^1\int_0^{1-\lambda} f\left(\lambda+\tau_2\right)(1-v)^{\alpha-1}v^{\beta-1}\tau_2^{\alpha+\beta-1}d\tau_2dv = \frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha+\beta)}\int_0^{1-\lambda} f\left(\lambda+\tau_2\right){\tau_2}^{\alpha+\beta-1}d\tau_2.$$

Hence

$$I = \frac{\Gamma\left(\alpha_1\right)\Gamma\left(\alpha_2\right)}{\Gamma\left(\alpha_1+\alpha_2\right)}\iint\ldots\int f\left(\tau_2+t_3+\ldots+t_n\right)\tau_2^{a_1+a_2-1}t_3^{a_3-1}\ldots t_n^{a_n-1}d\tau_2dt_3\ldots dt_n,$$

the integration being extended over all positive values of the variables such that $\tau_2+t_3+\ldots+t_n \leqslant 1$.

Continually reducing in this way we get

$$I = \frac{\Gamma\left(\alpha_1\right)\Gamma\left(\alpha_2\right)\ldots\Gamma\left(\alpha_n\right)}{\Gamma\left(\alpha_1+\alpha_2+\ldots+\alpha_n\right)}\int_0^1 f(\tau)\tau^{\Sigma_{i=1}^n\alpha_i-1}d\tau,$$

which is Dirichlet's result, formally

[1]The reference for this section is page 258, Whittaker, Edmund T., and George Neville Watson. A course of modern analysis: an introduction to the general theory of infinite processes and of analytic functions; with an account of the principal transcendental functions. University press, 1920.

Theorem 4.1

$$\int_0^1\int_0^{1-t_n}\cdots\int_0^{1-t_3-t_4-\cdots-t_n}\int_0^{1-t_2-t_3-\cdots-t_n} f\left(\sum_{i=1}^n t_i\right) t_1^{\alpha_1-1}t_2^{\alpha_2-1}\ldots t_n^{\alpha_n-1}dt_1dt_2\ldots dt_{n-1}dt_n = \\ = \frac{\Gamma(\alpha_1)\Gamma(\alpha_2)\ldots\Gamma(\alpha_n)}{\Gamma(\alpha_1+\alpha_2+\ldots+\alpha_n)}\int_0^1 f(\tau)\tau^{\Sigma_{i=1}^n\alpha_i-1}d\tau$$

♡

4.3 Volume of spherical type domain

Recall that the domain is defined as

$$D(a_1,a_2,\cdots,a_n):=\{(x_1,\cdots,x_n)\in\mathbb{R}^n|x_1^{a_1}+\cdots+x_n^{a_n}\le 1,\text{ and } x_i\ge 0,\forall i=1,2,\cdots,n\}.$$

So its volume is

$$\int_{D(a_1,a_2,\cdots,a_n)} 1dV =$$

$$= \int_0^1\int_0^{(1-x_n^{a_n})^{\frac{1}{a_{n-1}}}}\cdots\int_0^{(1-x_3^{a_3}-x_4^{a_4}-\cdots-x_n^{a_n})^{\frac{1}{a_2}}}\int_0^{(1-x_2^{a_2}-x_3^{a_3}-\cdots-x_n^{a_n})^{\frac{1}{a_1}}} 1dx_1dx_2\ldots dx_{n-1}dx_n \tag{4.3}$$

Let $t_i=x_i^{a_i}, i=1,\cdots,n$. By change of variables fomula, we have,

$$(4.3)=\frac{1}{a_1a_2\cdots a_n}\int_0^1\int_0^{1-t_n}\cdots\int_0^{1-t_3-t_4-\cdots-t_n}\int_0^{1-t_2-t_3-\cdots-t_n} t_1^{\frac{1}{a_1}-1}t_2^{\frac{1}{a_2}-1}\ldots t_n^{\frac{1}{a_n}-1}dt_1dt_2\ldots dt_{n-1}dt_n$$

This is the special case of theorem 4.1 for $f=1,\alpha_i=\frac{1}{a_i}$, so

$$\begin{aligned}(4.3) &= \frac{1}{a_1a_2\cdots a_n}\frac{\Gamma\left(\frac{1}{a_1}\right)\Gamma\left(\frac{1}{a_2}\right)\ldots\Gamma\left(\frac{1}{a_n}\right)}{\Gamma\left(\frac{1}{a_1}+\frac{1}{a_2}+\ldots+\frac{1}{a_n}\right)}\int_0^1\tau^{\Sigma_{i=1}^n\frac{1}{a_i}-1}d\tau \\ &= \frac{1}{a_1a_2\cdots a_n}\frac{\Gamma\left(\frac{1}{a_1}\right)\Gamma\left(\frac{1}{a_2}\right)\ldots\Gamma\left(\frac{1}{a_n}\right)}{\Gamma\left(\frac{1}{a_1}+\frac{1}{a_2}+\ldots+\frac{1}{a_n}\right)}\frac{1}{\Sigma_{i=1}^n\frac{1}{a_i}} \\ &= \frac{\Gamma\left(\frac{1}{a_1}+1\right)\Gamma\left(\frac{1}{a_2}+1\right)\ldots\Gamma\left(\frac{1}{a_n}+1\right)}{\Gamma\left(\frac{1}{a_1}+\frac{1}{a_2}+\ldots+\frac{1}{a_n}+1\right)}\end{aligned}$$

So we get

Theorem 4.2

$$V(a_1,a_2,\cdots,a_n)=\frac{\Gamma\left(\frac{1}{a_1}+1\right)\Gamma\left(\frac{1}{a_2}+1\right)\ldots\Gamma\left(\frac{1}{a_n}+1\right)}{\Gamma\left(\frac{1}{a_1}+\frac{1}{a_2}+\ldots+\frac{1}{a_n}+1\right)}$$

♡

The relative advantage of using Gamma function to denote the volume is that we have the Gamma function calculator, for example https://keisan.casio.com/exec/system/1180573444. So we are able to calculate by hands approximately the volume of any spherical type domain $D(a_1,a_2,\cdots,a_n)$ for $a_i>0$.

And for some particular a_i we could obtain the exact value easily.

Example 4.1 For positive integers m,n,l, the volume of the domain $D(m,n,l)$ is $\frac{\Gamma(m+1)\Gamma(n+1)\Gamma(n+1)}{\Gamma(m+n+l+1)}=\frac{m!n!l!}{(m+n+l)!}$

Example 4.2 Consider the domain D defined by $\{(x,y)|x^{\frac{2}{3}}+y^{\frac{2}{3}}\le 1\}$. Then its area is $4V(\frac{2}{3},\frac{2}{3})=$

$\frac{\Gamma(\frac{3}{2}+1)\Gamma(\frac{3}{2}+1)}{\Gamma(\frac{3}{2}+\frac{3}{2}+1)} = \frac{(\frac{3}{4}\sqrt{\pi})^2}{3!} = \frac{3}{32}\pi$ by (4.2). Also you can find the volume of the domain defined by $\{(x, y, z)|x^{\frac{2}{3}} + y^{\frac{2}{3}} + z^{\frac{2}{3}} \leq 1\}$. In fact for positive integers m, n, l, the volume of $\{(x, y, z)|x^{\frac{2}{m}} + y^{\frac{2}{n}} + z^{\frac{2}{l}} \leq 1\}$ is handleable using theorem 4.2.